

On the automorphism group of a Johnson graph

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Abstract

The Johnson graph $J(n, i)$ is defined to be the graph whose vertex set is the set of all i -element subsets of $\{1, \dots, n\}$, and two vertices are joined whenever the cardinality of their intersection is equal to $i - 1$. In Ramras and Donovan [*SIAM J. Discrete Math.*, 25(1): 267-270, 2011], it is conjectured that if $n = 2i$, then the automorphism group of the Johnson graph $J(n, i)$ is $S_n \times \langle T \rangle$, where T is the complementation map $A \mapsto \{1, \dots, n\} \setminus A$. We resolve this conjecture in the affirmative. The proof uses only elementary group theory and is based on an analysis of the clique structure of the graph.

Index terms — Johnson graph, automorphism group, cliques

1. Introduction

The Johnson graph $J(n, i)$ is defined to be the graph whose vertex set is the set of all i -element subsets of $\{1, \dots, n\}$, and two vertices A, B are said to be adjacent in this graph whenever $|A \cap B| = i - 1$. This graph has been well-studied in the literature (cf. [1] [2] [3] [4] [5] [8] [9] [10]). The automorphism group of a graph is the set of all permutations of the vertex set of the graph that preserves adjacency [6]. In [10], it is proved that if $n \neq 2i$, then the automorphism group of the Johnson graph $J(n, i)$ is isomorphic to S_n . In [10, Conjecture 1, p. 269] it is conjectured that if $n = 2i$, then the automorphism group of $J(n, i)$ is isomorphic to $S_n \times \langle T \rangle$, where T is the complementation map $A \mapsto A^c$ and $A^c := \{1, \dots, n\} \setminus A$. In the present paper, this conjecture is resolved in the affirmative.

Actually, the automorphism group of $J(n, i)$ for both the $n \neq 2i$ and $n = 2i$ cases was already determined in [7], but the proof given there uses heavy group-theoretic machinery. The main result of [10] was to provide a proof for the $n \neq 2i$ case that uses only elementary group theory; the proof is based on an analysis of the clique structure of the graph. In [10] the authors leave the $n = 2i$ case open but make a conjecture for this case. We resolve this conjecture in the affirmative by providing a proof that again uses only elementary group theory and a similar analysis of the clique structure of the graph.

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We first recall some basic facts about the Johnson graphs $J(n, i)$. Two vertices A, B are adjacent in this graph iff their intersection $A \cap B$ has cardinality $i - 1$, and this occurs exactly when the cardinality of their symmetric difference is 2. The Johnson graph $J(n, i)$ is isomorphic to the Johnson graph $J(n, n - i)$; an explicit bijection between their vertex sets that preserves adjacency is the complementation map $T : A \mapsto A^c$. Hence, without loss of generality we shall restrict our study of the Johnson graphs $J(n, i)$ to the case where $i \leq n/2$. Also, the graphs $J(n, 1)$ are the complete graphs and hence are not very interesting. The graphs $J(n, 2)$ are the line graphs of complete graphs, and their automorphism groups are known. Thus, when studying $J(n, i)$ henceforth, it is assumed that $i \geq 3$.

Each permutation in S_n acts in a natural way on the set of i -element subsets of $\{1, \dots, n\}$, and this induced action on the vertices of $J(n, i)$ is an automorphism of the graph. Also, distinct permutations in S_n induce distinct permutations of the i -element subsets. Hence S_n is isomorphic to a subgroup of the automorphism group of $J(n, i)$. In some cases, S_n happens to be the (full) automorphism group of $J(n, i)$. A special case of the results in [7, Theorem 2(a)(c)] is that when $n \neq 2i$, the automorphism group of $J(n, i)$ is isomorphic to S_n ; a special case of the results in [7, Theorem 2(e)] is that when $n = 2i$, the automorphism group of $J(n, i)$ is isomorphic to $S_n \times S_2$. The proofs given in [7] use heavy group-theoretic machinery. An elementary combinatorial proof of the former result is given in [10], and an elementary combinatorial proof of the latter result is given in the present paper.

The following is the main result proved in the present paper:

Theorem 1. *If $n = 2i$, then the automorphism group of the Johnson graph $J(n, i)$ is $S_n \times \langle T \rangle$, where T is the complementation map $A \mapsto A^c$.*

For $\theta \in S_n$, let ρ_θ denote the permutation of the vertex set of $J(n, i)$ induced by θ . It is clear that $\{\rho_\theta : \theta \in S_n\}$ is a subgroup of the automorphism group of $J(n, i)$. When $n = 2i$, the subgroup $\langle T \rangle$ also acts as a group of automorphisms of $J(n, i)$:

Lemma 2. *Suppose $n = 2i$. Then the complementation map $T : A \mapsto A^c$ is an automorphism of the Johnson graph $J(n, i)$.*

Proof: Let A and B be two vertices in $J(n, i)$. We show that A and B are adjacent in $J(n, i)$ iff A^c and B^c are adjacent in $J(n, i)$. Recall that two vertices are adjacent in $J(n, i)$ iff their intersection has cardinality $i - 1$. The cardinality $|A^c \cap B^c| = n - |A \cup B| = n - (|A| + |B| - |A \cap B|) = n - 2i + |A \cap B|$, which equals $|A \cap B|$ since $n = 2i$. Since $A \cap B$ and $A^c \cap B^c$ have the same cardinality, the complementation map preserves adjacency and nonadjacency in $J(n, i)$. \blacksquare

The group $\{\rho_\theta : \theta \in S_n\} \langle T \rangle$ of automorphisms of $J(n, i)$ obtained so far can be expressed as a direct product:

Lemma 3. *Let T denote the complementation map $A \mapsto A^c$. The group $H := \{\rho_\theta : \theta \in S_n\} \langle T \rangle$ of automorphisms of $J(2i, i)$ is isomorphic to the direct product $S_n \times \langle T \rangle \cong S_n \times S_2$.*

Proof: Observe that if A is any i -element subset of $\{1, \dots, n\}$, then $[\theta(A)]^c = \theta(A^c)$, whence T and ρ_θ commute. It follows that $\{\rho_\theta : \theta \in S_n\} \langle T \rangle$ is a group and its two factors are normal subgroups. It remains to show that the two factors $\{\rho_\theta : \theta \in S_n\}$ and $\langle T \rangle$ have a trivial intersection. By way of contradiction, suppose $T = \rho_\theta$ for some $\theta \in S_n$. Then θ takes $\{1, \dots, i-1, i\}$ to its complement $\{i+1, \dots, 2i\}$, and $\{1, \dots, i-1, i+1\}$ to its complement $\{i, i+2, \dots, 2i\}$. Hence θ takes the common elements $\{1, \dots, i-1\}$ to $\{i+2, \dots, 2i\}$, and hence the remaining elements $\{i, i+1\}$ to $\{i, i+1\}$. Take $A = \{2, \dots, i-1, i, i+1\}$. Then $A^{\rho_\theta} \supseteq \{i, i+1\}$. Thus $A^{\rho_\theta} \neq A^c$, a contradiction. \blacksquare

Notation. Fix a vertex X of the graph $J(n, i)$. Let \mathcal{L}_i denote the set of vertices of $J(n, i)$ whose distance to X is exactly i . Thus, $\mathcal{L}_0 = \{X\}$, and \mathcal{L}_1 is the set $N(X)$ of neighbors of X . Let G denote the automorphism group of $J(n, i)$. The stabilizer of X in G is denoted G_X .

We use the following additional notation from [10]. Each neighbor of a vertex X in $J(n, i)$ is of the form $(X - \{p\}) \cup \{q\}$ for some $p \in X, q \notin X$. We denote this neighbor by $Y_{p,q}$. For each $p \in X$, the set of neighbors $\{Y_{p,q} : q \notin X\}$ forms a clique, denoted by \mathcal{Y}_p . The set $\{\mathcal{Y}_p : p \in X\}$ is a partition of the set $N(X)$ of neighbors of X into i cliques, each of cardinality $n - i$. Similarly, for each $q \notin X$, the set $\{Y_{p,q} : p \in X\}$ forms a clique, denoted by \mathcal{Z}_q . The set $\{\mathcal{Z}_q : q \notin X\}$ is a partition of $N(X)$ into $n - i$ cliques, each of cardinality i . Each maximal clique in $J(n, i)$ that contains the vertex X is of the form $\{X\} \cup \mathcal{Y}_p$ for some $p \in X$ or of the form $\{X\} \cup \mathcal{Z}_q$ for some $q \notin X$ (cf. [10, Lemma 1]).

We call each clique \mathcal{Y}_p a clique of the first kind. Similarly, each clique \mathcal{Z}_q is a clique of the second kind. When $n \neq 2i$, the cardinality of a clique of the first kind is not equal to the cardinality of a clique of the second kind; thus, any automorphism of the graph that fixes the vertex X must permute the set of cliques of the first kind in $N(X)$ amongst themselves. On the other hand, when $n = 2i$, the cliques in $N(X)$ of the first and second kind have the same cardinality, and so it is possible that there is an automorphism in G_X that takes a clique of the first kind to a clique of the second kind. Indeed, we show below that such an automorphism exists and can be expressed in terms of the complementation map.

Proposition 4. *Suppose $n = 2i$, and let X be a vertex of $J(n, i)$ and let $g \in G_X$. Then there exist $\theta \in S_n$ and $i \in \{0, 1\}$ such that the actions of g and $\rho_\theta T^i$ on $\mathcal{L}_0 \cup \mathcal{L}_1$ are identical.*

Proof: Let $g \in G_X$. Then g acts on the set $N(X)$ of neighbors of X , and hence permutes the maximal cliques in $N(X)$ amongst themselves. Recall that these maximal cliques are either of the first kind or the second kind. We consider two cases.

First suppose that g permutes the set of cliques in $N(X)$ of the first kind amongst themselves. Since $g \in G_X$, g acts bijectively on the set of all maximal cliques in $N(X)$, and so g also permutes the set of cliques of the second kind amongst themselves. Hence $g : \mathcal{Y}_p \mapsto \mathcal{Y}_{\theta_1(p)}, \mathcal{Z}_q \mapsto \mathcal{Z}_{\theta_2(q)}$ for some $\theta_1 \in \text{Sym}(X), \theta_2 \in \text{Sym}(X^c)$. Define $\theta \in S_n$ to be the map that takes j to $\theta_1(j)$ if $j \in X$ and that takes j to $\theta_2(j)$ if $j \in X^c$. As shown in [10, p. 268], the actions of g and ρ_θ on $\mathcal{L}_0 \cup \mathcal{L}_1$ are identical.

For the rest of the proof, suppose that g takes some clique of the first kind to a clique of the second kind. So there exist $p' \in X, q' \notin X$ such that $g : \mathcal{Y}_{p'} \mapsto \mathcal{Z}_{q'}$. We show that g takes every clique of the first kind to some clique of the second kind. Observe that $\mathcal{Z}_{q'}$ contains exactly one vertex from \mathcal{Y}_p , for each $p \in X$. Any two cliques of the first kind are disjoint, and g must map disjoint cliques to disjoint cliques. Also, any two cliques of the second kind are disjoint, whereas a clique of the first kind and a clique of the second kind meet: $\mathcal{Y}_p \cap \mathcal{Z}_q \neq \emptyset$ since it contains $Y_{p,q}$. Thus, if g takes a clique of the first kind to a clique of the second kind, then g takes each clique of the first kind to some clique of the second kind. Hence g interchanges the set of cliques of the first kind and the set of cliques of the second kind.

Thus $g : \mathcal{Y}_p \mapsto \mathcal{Z}_{\theta_1(p)}, \mathcal{Z}_q \mapsto \mathcal{Y}_{\theta_2(q)}$ for some $\theta_1 : X \mapsto X^c$ and $\theta_2 : X^c \mapsto X$. Define $\theta \in S_n$ to be the map that takes j to $\theta_1(j)$ if $j \in X$ and that takes j to $\theta_2(j)$ if $j \in X^c$. Recall that ρ_θ is defined as the action of θ induced on the vertex set of $J(n, i)$ and that T denotes the complementation map $A \mapsto A^c$.

We show that the actions of g and $\rho_\theta T$ on $\mathcal{L}_0 \cup \mathcal{L}_1$ are identical. It is clear that both the actions fix $\mathcal{L}_0 = \{X\}$. For $g \in G_X$ implies g fixes X . And $X^{\rho_\theta T} = (X^c)^T = X$. Let $Y_{p,q}$ be a vertex in \mathcal{L}_1 , and consider the action of g and $\rho_\theta T$ on this vertex. Recall that $Y_{p,q}$ is the unique vertex in the intersection $\mathcal{Y}_p \cap \mathcal{Z}_q$. We have that $(\mathcal{Y}_p \cap \mathcal{Z}_q)^g = \mathcal{Z}_{\theta_1(p)} \cap \mathcal{Y}_{\theta_2(q)} = Y_{\theta_2(q), \theta_1(p)}$. The vertex $Y_{p,q}$ has the same image under $\rho_\theta T$ as under g : $(Y_{p,q})^{\rho_\theta T} = ((X - \{p\}) \cup \{q\})^{\rho_\theta T} = [(X^{\theta_1} - \{\theta_1(p)\}) \cup \{\theta_2(q)\}]^T = [(X^c - \{\theta_1(p)\}) \cup \{\theta_2(q)\}]^T = (X - \{\theta_2(q)\}) \cup \{\theta_1(p)\} = Y_{\theta_2(q), \theta_1(p)}$. Thus, g and $\rho_\theta T$ act identically on \mathcal{L}_1 . \blacksquare

The following result, which is proved in [10, Lemma 2 and Proposition 1], does not use the condition that $n \neq 2i$ and hence also applies when $n = 2i$:

Lemma 5. *In the Johnson graph $J(n, i)$, if an automorphism g fixes a vertex X and each of its neighbors, then it is the trivial automorphism.*

We now complete the proof of the main theorem.

Corollary 6. *If $n = 2i$, then the automorphism group of the Johnson graph $J(n, i)$ is $S_n \times \langle T \rangle$, where T is the complementation map $X \mapsto X^c$.*

Proof: Let $g \in G_X$. By Proposition 4, there exist $\theta \in S_n$ and $i \in \{0, 1\}$ such that the action of g and $\rho_\theta T^i$ are identical on $\mathcal{L}_0 \cup \mathcal{L}_1$. Hence, $g^{-1} \rho_\theta T^i$ acts trivially on $\mathcal{L}_0 \cup \mathcal{L}_1$. By Lemma 5, $g^{-1} \rho_\theta T^i$ is the trivial automorphism of $J(n, i)$. Hence $g = \rho_\theta T^i$. This proves that every element in G_X is one of the $2(i!)^2$ automorphisms specified in the proof above, i.e. every element in G_X is either one of the $i!i!$ elements in G_X that permutes the i cliques of the first kind amongst themselves and the i cliques of the second kind amongst themselves, or is one of the $i!i!$ elements in G_X that interchanges the set of cliques of the first kind and the set of cliques of the second kind. Hence $|G_X| = 2(i!)^2$. Finally, since the graph $J(n, i)$ is vertex-transitive, the automorphism group G has order $|G_X| \binom{n}{i} = 2(i!)^2 \binom{n}{i} = 2n!$. Hence the group of automorphisms $S_n \times \langle T \rangle$ obtained above is the (full) automorphism group of $J(n, i)$. \blacksquare

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